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PROGRAM A - REVISIONS

~~Technical Memorandum~~

CDC-TM-9552-3

Feasibility Study of a
Track-While-Scan Navigation Concept

(NASA Contract No. NAS1-2902)

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Prepared by:

Joseph E. Carroll

Research Division
Control Data Corporation
3101 East 80th Street
Minneapolis 20, Minnesota

0671222

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PROGRAM A REVISIONS

Introduction and Summary

Since the publication of "Error Analysis - Program A"* a number of revisions have taken place, some minor, others not so minor. This memorandum will detail these revisions and may generally be taken as the final descriptive report of Program A.**

As will be recalled, Program A effectively simulates the scanning camera navigation device in that randomized errors are introduced into the measurements, and position and attitude error patterns calculated from these measurements are studied. In TM-9552-2 an extensive description of the program was undertaken together with derivations of some of the pertinent equations. A number of minor errors or revisions occurred there and these are corrected here.

Of major importance, however, is the slight change in the navigational equations. Actually it is a change in philosophy which hinges upon which quantity to minimize in the least squares analysis. Previously the residuals of a certain function were minimized. It seems wiser, on subsequent thinking, to minimize the residuals of the measured quantity. This is detailed in the body together with the necessary equations.

Some improvements were made in the statistical analysis section and these are also given herein, though some question can be raised as to their information content, especially concerning the chi-squared test.

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* Carroll, J.E., "Error Analysis - Program A", CDC TM-9552-2, July 29, 1963. (Hereafter referred to as TM-9552-2 in the text.)

** Memoranda will of course be forthcoming on the actual program usage and also on various error analysis results.

A. Minor Revisions

Table I lists those errata for TM-9552-2 which can be considered as minor errors in the memorandum itself or revisions to Program A. Most important are the following:

(a) The obliquity of the ecliptic is taken at 1970.0 and is equal to $23^{\circ}26'35''.47$.

(b) Instead of the fifty brightest stars, 100 are chosen with the capability in the program of specifying how many are to be used. These stars with their 1970.0 equatorial coordinates are listed in Table II. Right ascension and declination were found by simple linear extrapolation from the 1950.0 positions in Atlas Coeli using Annual Variation in each coordinate.

(c) In generating random gaussian transit time errors, the $F_N(x)$ table was extended to ± 3.4 sigma, thereby utilizing 99.94% of all cases, or omitting only 3 out of every 5,000.

(d) One of the important partial derivatives had been omitted from the Appendix of TM-9552-2. It is given in Table I.

TABLE I: Minor Revisions in TM-9552-2 and in Program A

Page No.	Line	Change
abstract	par. d	"the 100 brightest..."
3	24	"100 brightest stars with..."
10	Eq. (10)	change - to + before arcsine
12	last	"is the obliquity of the 1970.0 ecliptic which we take as"
13	first	"$= 23^\circ 26' 35''.47.$(6) In..."
13	ftnte	Insert: (6) Allen, C. W., <u>Astrophysical Quantities</u>, Athlone Press, 1963, p. 18.
14		This table replaced by 100 brightest stars as given in Allen, <u>Astrophysical Quantities</u>, 1963, p. 229. (See Table II of the present memorandum.) Reference (7) also changes, therefore.
15	Eq. (18)	$\lambda_{o\ell}$ should be given in radians.
15	6	"and $G = 1.996 \times 10^{-47}$ $(A.U.)^3/sec^2$ gm is...."
17	1	"as RNGN. This...."
17	5-6	"...from -3.4 to +3.4 in increments..."
18	1-4	Since $-3.4 \lesssim t \lesssim +3.4$, then $-3.4\sigma \leq \Delta q \leq +3.4\sigma$; i.e., the assumption is made that we are only dealing with 99.94% of the possible cases. This is certainly not a serious restriction, however.
22	Eq. (29)	- sign before M
B-2	Eq. (B-3)	Change - to + before A
B-3		After Eq. (B-8) and before the line: "where A and B...", insert: $\frac{\partial f}{\partial t_0} = A \frac{2\pi}{T} \cos\Psi - B \frac{2\pi}{T} \sin\Psi.$ (B-9) Equation numbers (B-9) - (B-11) then become (B-10) - (B-12).

TABLE II: 100 BRIGHTEST STARS*

GENERAL CATALOGUE NUMBER	RIGHT ASCENSION** (1970.0 EQUATOR)			DECLINATION** (1970.0 EQUATOR)			VISUAL MAGNITUDE
	h	m	s	o	'	"	
8833	6.	43.	49.5	-16.	*40.	*24.	-1.44
8302	6.	23.	17.1	-52.	*40.	*44.	-.72
19728	14.	37.	32.8	-60.	*42.	*47.	-.27
19242	14.	14.	17.6	19.	20.	17.	-.05
25466	18.	35.	55.3	38.	45.	17.	.03
6427	5.	14.	28.1	45.	58.	12.	.09
6410	5.	13.	5.6	-8.	*14.	-5.	.11
10277	7.	37.	43.9	5.	18.	12.	.36
1979	1.	36.	35.8	-57.	*23.	*19.	.49
18971	14.	1.	41.3	-60.	*13.	*46.	.63
7451	5.	53.	32.8	7.	24.	12.	.40
27470	19.	49.	19.2	8.	47.	15.	.77
5605	4.	34.	11.7	16.	27.	1.	.80
16952	12.	24.	54.8	-62.	*56.	-.	.83
22157	16.	27.	33.8	-26.	*22.	-2.	.94
18144	13.	23.	36.5	-11.	-.	*20.	.97
32000	22.	55.	59.7	-29.	*46.	*54.	1.16
10438	7.	43.	29.0	28.	6.	1.	1.15
28846	20.	40.	24.3	45.	10.	21.	1.25
17374	12.	45.	57.3	-59.	*31.	*31.	1.29
13926	10.	6.	46.5	12.	6.	52.	1.34
9188	6.	57.	26.8	-28.	*55.	*48.	1.48
10120	7.	32.	41.3	31.	57.	20.	1.56
23769	17.	31.	34.0	-37.	-5.	-2.	1.60
6668	5.	23.	31.3	6.	19.	28.	1.63
6681	5.	24.	23.6	28.	35.	2.	1.65
12764	9.	12.	52.9	-69.	*35.	*36.	1.68
17052	12.	29.	29.3	-56.	*56.	*45.	1.68
6960	5.	34.	41.3	-1.	*13.	*10.	1.70
30942	22.	6.	21.0	-47.	-6.	*27.	1.75
17518	12.	52.	42.9	56.	7.	21.	1.78
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
4041	3.	22.	10.2	49.	45.	22.	1.80
25100	18.	22.	10.8	-34.	*24.	-3.	1.81
15185	11.	1.	53.5	61.	54.	49.	1.81
11105	8.	8.	36.5	-47.	*14.	*50.	1.85
9443	7.	7.	10.3	-26.	*20.	*39.	1.85
18643	13.	46.	21.5	49.	27.	44.	1.86
23857	17.	35.	9.6	-42.	*58.	*51.	1.86
7543	5.	57.	19.6	44.	56.	49.	1.89
8633	6.	35.	58.8	16.	25.	35.	1.93
12069	8.	43.	52.6	-54.	*35.	*53.	1.93
22558	16.	45.	28.2	-68.	*58.	*32.	1.93
28374	20.	23.	17.0	-56.	*50.	-.	1.94
11463	8.	21.	54.0	-59.	*24.	*45.	1.94

* Allen, C. W., Astrophysical Quantities, Athlone Press, 1963, p. 229.

** Computed by linear extrapolation from 1950.0 coordinate (Atlas Coeli, 1960) using Annual Variation.

GENERAL CATALOGUE NUMBER	RIGHT ASCENSION (1970.0 EQUATOR) h	m	s	DECLINATION (1970.0 EQUATOR) °	'	"	VISUAL MAGNITUDE
8223	6	21	22.6	-17	*56	*23	1,96
2538	2	5	28.5	23	19	17	2,00
2796	2	17	49.6	-3	-6	*47	2,00
2243	1	50	7.1	89	7	40	2,02
14177	10	18	19.3	19	59	39	2,02
865		42	5.0	-18	-9	-3	2,04
20029	14	50	46.3	74	16	41	2,04
13044	9	26	6.8	-8	*31	*39	2,05
7264	5	46	20.0	-9	*40	*43	2,06
127		6	49.8	28	55	30	2,07
23837	17	33	32.3	12	34	50	2,07
19033	14	4	54.5	-36	*13	*24	2,07
1400	1	8	2.7	35	27	44	2,07
25941	18	53	24.3	-26	*20	-9	2,09
16953	12	24	55.5	-62	*56	-2	2,09
3733	3	6	12.6	40	50	30	2,10
18133	13	22	43.4	55	4	53	2,12
16189	11	47	31.8	14	44	24	2,13
1117		54	52.7	60	33	17	2,15
17262	12	39	51.1	-48	*47	*44	2,16
2477	2	2	2.8	42	11	13	2,16
31685	22	40	53.0	-47	-2	*32	2,16
6847	5	30	28.4		20	58	2,19
792		38	47.5	56	22	23	2,20
24432	17	55	54.4	51	29	30	2,21
20947	15	33	25.0	26	48	52	2,22
28338	20	21	8.9	40	9	34	2,22
10947	8	2	31.8	-39	*55	-3	2,23
12623	9	6	53.5	-43	*18	*38	2,23
12831	9	16	17.3	-59	-8	*56	2,24
147		7	33.8	58	59	5	2,26
22640	16	48	13.0	-34	*14	*26	2,29
21489	15	58	33.3	-22	*32	*16	2,32
18458	13	37	58.3	-53	*18	*53	2,34
[illegible]	[illegible]		[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
519		24	48.2	-42	*28	-9	2,37
30431	21	42	42.7	9	44	11	2,38
23988	17	40	24.5	-39	-1	-1	2,39
19656	14	33	35.5	-42	-1	*38	2,39
19856	14	43	40.6	27	11	58	2,39
9886	7	22	54.4	-29	*14	*36	2,42
16268	11	52	15.6	53	51	42	2,43
29848	21	17	51.8	62	27	30	2,43
23158	17	8	39.3	-15	*41	*23	2,44
12938	9	21	11.1	-54	*52	*56	2[illegible]
28959	20	44	59.8	33	51	25	2,46
32149	23	3	15.9	15	2	37	2,49
19774	14	39	55.3	-47	*15	*40	2,50
32135	23	2	19.0	27	55	10	2,50
21609	16	3	41.3	-19	*43	*31	2,52
3643	3		42.6	3	58	23	2,53
15438	11	12	30.9	20	41	18	2,55
22332	16	35	30.1	-10	*30	*29	2,56
26161	19		42.3	-29	*55	*31	2,57
16740	12	14	15.7	-17	*22	*32	2,58

B. Least Squares Navigation Computation

In TM-9552-2, the minimum data attitude and position computations are correct as given. The least squares method, however, rests on a precarious assumption and therefore must be rescued. Furthermore, it is unnecessarily complicated in that both attitude and position are computed together. They can, since stars contribute no information towards vehicle position, be evaluated separately, greatly relieving thereby the amount of computation required.

The assumption mentioned above and the ensuing problem concerns itself with the non-linearity of the function connecting the measured transit time and the vehicle unknowns. The least squares analysis developed in the Appendix to TM-9552-2 was based on minimizing the sum of the squares of the residuals of this function. When the measured value is buried in a nonlinear manner in the function, this is not in general the procedure for computing the "best" values of the unknowns. Physically, one expects that if the measured values are believed to be normally distributed about the true value as mean, then the values of the computed unknowns should cluster about a value corresponding to that mean. By minimizing the sum of the squares of the function residuals, one is finding values of the unknowns which correspond to the mean of the function residuals, and these are not necessarily the same values as would correspond to the mean of the measured values. Said in other words, one certainly acts correctly in minimizing the sum of the squares of the residuals of the measured quantities, but there can arise some question as to the validity of this result when the same procedure is applied to the residuals of a function connecting the measured and unknown quantities.

Thus we reformulate the least squares navigation computation on the following lines. Let $f_k(x_1, x_2, \ldots x_n; t_k) = 0$ represent the function connecting the measurable parameter t_k and the unknowns x_j, $j = 1, 2, \ldots, n$. Because of the non-linearity of f_k in the variable t_k and perhaps in some of the unknowns, we first linearize the function using assumed values x_j' and t_k':

$$f_k' + \sum_{j=1}^{n} \left(\frac{\partial f_k}{\partial x_j}\right)' \Delta x_j + \left(\frac{\partial f_k}{\partial t_k}\right)' \Delta t_k = 0 \qquad (1)$$

where the primes denote evaluation at x_j' and t_k' and the true values are $x_j = x_j' + \Delta x_j$, $t_k = t_k' + \Delta t_k$. Let p measurements of t_k now be made, one measurement for each $k(k = 1, 2, \ldots, p)$, yielding values t_{km}. The expectation of any of these values is just the solution of (1); i.e., a linear function of the incremental unknowns Δx_j:

$$\mathcal{E}(t_{km}) = t_k' - \frac{f_k' + \sum_{j=1}^{n} \left(\frac{\partial f_k}{\partial x_j}\right)' \Delta x_j}{\left(\frac{\partial f_k}{\partial t_k}\right)'} . \qquad (2)$$

The Gauss-Markoff theorem* states that the best unbiased linear least squares estimates Δx_j are those values of the Δx_j that minimize the sum of squares

$$S = \sum_{k=1}^{p} \left(t_{km} - \mathcal{E}(t_{km})\right)^2 w_k \qquad (3)$$

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* Trumpler and Weaver, Statistical Astronomy, Dover Press, New York, 1953, p. 182.

where w_k is a weighting function dependent on the variance σ_k^2 of t_{km}.

In our navigational problem, all of the measurements are assumed performed in the same manner with the same instrument on essentially identical targets.* Thus all of the w_k are the same and can be dropped from the notation. Furthermore, for lack of any better estimate, we let $t_k' = t_{km}$. That is, we evaluate the function and its partials at the measured value t_{km} since this represents a highly accurate approximation to the true t_k.

The equations from which the Δx_j are calculated come from minimizing S with respect to the unknowns:

$$\frac{\partial S}{\partial \Delta x_j} = 0 \quad , \quad j = 1, 2, \ldots, n. \tag{4}$$

Substituting t_{km} for t_k', Equation (2) in Equation (3), carrying out the differentiations, and rearranging terms, gives, finally, the solution

$$\widetilde{\Delta X} = M^{-1} B \tag{5}$$

where $\widetilde{\Delta X}$ is a n x 1 column vector whose elements are the Δx_j, M is an n x n square matrix with elements a_{ij}:

$$a_{ij} = \sum_{k=1}^{p} \frac{(\partial f_k/\partial x_i)'(\partial f_k/\partial x_j)'}{(\partial f_k/\partial t_k)'^2} \tag{6}$$

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* In a future program, the target measurement errors will differ depending on their magnitude.

and B is an n x 1 column vector whose elements are c_i:

$$c_i = -\sum_{k=1}^{p} \frac{f_k' (\partial f_k / \partial x_i)'}{(\partial f_k / \partial t_k)'^2} \, . \tag{7}$$

The difference between these coefficients and those in TM-9552-2 (Equations (40)-(41)) are that the former have $(\partial f_k/\partial t_k)'^2$ in the denominator, acting somewhat as a weighting function. This is only a slight modification while the philosophical outlook is quite different and, with the present analysis, more convincing.

In our navigational problem, the function f_k is linear in the position coordinates X, Y, Z of the vehicle, but nonlinear in the attitude parameters θ, ϕ, t_0 and also in the transit time t_k. In addition, the coefficients of the position coordinates are themselves nonlinear in the attitude parameters and transit time. We can thus write f_k as

$$\begin{aligned} f_k &= X\, a_k(\theta, \phi, t_0; t_k) + Y\, b_k(\theta, \phi, t_0; t_k) \\ &\quad + Z\, c_k(\theta, \phi, t_0; t_k) + d_k(\theta, \phi, t_0; t_k) = 0. \end{aligned} \tag{8}$$

From a set of transit time measurements (one measurement for each k) the position coordinates and attitude parameters are to be computed. This will be done separately; that is the attitude parameters will be computed using only star transit time data (here the f_k is <u>not</u> a function of X, Y, Z), while the position of the vehicle will be derived only from planet transit times. Neither will be assumed to affect the other.

This last point requires some discussion. In an actual navigation situation, the attitude, even though it be determined only by stars, is used in computing quantities for the position determination. That is, the vehicle attitude must be known before position can be found. Thus, any errors in attitude will probably contribute errors to the position.

We are more interested at the present time, however, in deducing how fundamental transit time errors affect attitude and position separately. At some later date, their mutual influences will be examined. Therefore, in Program A, we will compute the vehicle position using for the attitude its true values. (To date there is no similar dependence of attitude on position--only stars are used for attitude.)

The computation of attitude will thus proceed as follows. Equation (5) can be written as

$$\begin{pmatrix} \Delta\theta \\ \Delta\phi \\ \Delta t_o \end{pmatrix} = M^{-1} B \tag{9}$$

where M and B have their elements specified by Equations (6) and (7). The function and partial derivatives there shown are evaluated at the true values of the attitude specified at the start of the program and at the measured value of the transit time t_{km}. The primes denote such an evaluation. The explicit forms are shown in the Appendix to TM-9552-2, Equations (B-1)-(B-3)* and in Equations (25)-(28) in the body of the memorandum. (It will be noticed that $\partial f_k / \partial t_k = - \partial f_k / \partial t_o$.)

The computation of position is quite similar. At first sight a significant improvement might seem possible due to the linearity of the function in the position coordinates. This is not so because of the dependence of a, b, c in Equation (8) on t_k. Attempting to compute the

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* Equation (B-3) should have a plus sign in front of A.

position rather than the increments in position will thus result in a quadratic or worse in three unknowns.

To escape this complexity and still compute the position directly we will use assumed values of X, Y, Z for $\partial f_k / \partial t_k$. In the form (8) we note that $\partial f / \partial X = a$, $\partial f / \partial Y = b$, $\partial f / \partial Z = c$ so that on substituting into (6) and (7) the vehicle position becomes

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = M^{-1} B \tag{10}$$

where M is a symmetric matrix with elements

$$a_{11} = \sum_{k=1}^{p} \frac{a_k'^2}{(\partial f_k / \partial t_k)'^2} \tag{11}$$

$$a_{22} = \sum_{k=1}^{p} \frac{b_k'^2}{(\partial f_k / \partial t_k)'^2} \tag{12}$$

$$a_{33} = \sum_{k=1}^{p} \frac{c_k'^2}{(\partial f_k / \partial t_k)'^2} \tag{13}$$

$$a_{12} = \sum_{k=1}^{p} \frac{a_k' b_k'}{(\partial f_k / \partial t_k)'^2} \tag{14}$$

$$a_{13} = \sum_{k=1}^{p} \frac{a_k' c_k'}{(\partial f_k / \partial t_k)'^2} \tag{15}$$

$$a_{23} = \sum_{k=1}^{p} \frac{b_k' c_k'}{(\partial f_k / \partial t_k)'^2} \tag{16}$$

$$a_{ij} = a_{ji} \tag{17}$$

and B is a column vector with elements

$$c_1 = -\sum_{k=1}^{p} \frac{d_k' a_k'}{(\partial f_k / \partial t_k)'^2} \tag{18}$$

$$c_2 = -\sum_{k=1}^{p} \frac{d_k' b_k'}{(\partial f_k / \partial t_k)'^2} \tag{19}$$

$$c_3 = -\sum_{k=1}^{p} \frac{d_k' c_k'}{(\partial f_k / \partial t_k)'^2} . \tag{20}$$

The explicit forms are to be found in Equations (33)-(36) in TM-9552-2 and from Equation (B-9)* in Appendix B to that memorandum. (Again note that $\partial f_k / \partial t_k = -\partial f_k / \partial t_o$.) These matrix elements are all evaluated at the true values X, Y, Z, θ, ϕ, t_o and at the measured transit time t_{km}. Thus many of the quantities can be compute before

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* Equation (B-9) is identical to Equation (B-3) and should have been included immediately after (B-8) and just before the line of text: "where A and B are...". The Equations (B-9)-(B-11) become numbered (B-10)-(B-12).

the Monte Carlo process in Program A leaving only simple manipulations after the measured time t_{km} becomes known. This is also true for attitude, of course.

C. Statistical Analysis

The method of analyzing the list of errors in any of the six attitude and position values has been altered to the extent of computing additional moments of the distribution, plotting the distribution, and subjecting it to a chi-squared test.

For computation of distribution moments, let each entry be represented by x_i. Then

$$\nu_r = \frac{1}{n} \sum_{i=1}^{n} x_i^r \tag{11}$$

are computed for r = 1, 2, 3, and 4. These are the first four moments of the distribution about the origin.* They are related to the central moments (moments about the mean value $\bar{x}$), μ_r, through

$$\left.\begin{aligned} \mu_1 &= 0 \\ \mu_2 &= \nu_2 - \nu_1^2 = \sigma^2 \\ \mu_3 &= \nu_3 - 3\nu_2\nu_1 + 2\nu_1^3 \\ \mu_4 &= \nu_4 - 4\nu_3\nu_1 + 6\nu_2\nu_1^2 - 3\nu_1^4 \end{aligned}\right\} \tag{12}$$

and

$$\left.\begin{aligned} \nu_1 &= \bar{x} \\ \nu_2 &= \mu_2 + \nu_1^2 \\ \nu_3 &= \mu_3 + 3\mu_2\nu_1 + \nu_1^3 \\ \nu_4 &= \mu_4 + 4\mu_3\nu_1 + 6\mu_2\nu_1^2 + \nu_1^4 \end{aligned}\right\} \tag{13}$$

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* Burington and May, Handbook of Probability and Statistics with Tables, Handbook Publishers Inc., Sandusky, Ohio, 1958, p. 9, 10.

Also computed are: (a) the coefficient of skewness*

$$\gamma_1 = \mu_3 / \sigma^3 \tag{14}$$

which indicates the asymmetry of the distribution about the mean ($\gamma_1 > 0$ indicates a "long tail" to the right while $\gamma_1 < 0$ indicates a similar feature on the left) and (b) the coefficient of excess*

$$\gamma_2 = (\mu_4 / \sigma^4) - 3 \tag{15}$$

which is a measure of the degree of "flattening" (kurtosis) of the distribution near its center.

The computation of these moments and coefficients will define the sample distribution quite well, since, for a normal distribution about mean zero,

$$\left.\begin{aligned} &\mu_r = \nu_r, \qquad \gamma_1 = \gamma_2 = 0 \\ &\mu = \mu = \mu = \cdots = 0 \\ \text{and} \quad &\mu_{2r} = (2r-1)!!\,\sigma^{2r} \end{aligned}\right\} \tag{16}$$

Checking these against the sample values will clearly show the extent to which the individual errors are normally distributed.

For visual analysis, a plot is generated of the distribution of points. It is presented along with the listing of the cumulative distribution. (See Appendix A of TM-9552-4 for a typical print-out.)

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* Burington and May, Handbook of Probability and Statistics with Tables, Handbook Publishers Inc., Sandusky, Ohio, 1958, p. 13, 23, 24.

To further accumulate data concerning each distribution, a chi-squared test is performed on the fit of the sample to a normal distribution.* Having already found the sample mean $\bar{x}$ and standard deviation σ, and having generated a cumulative distribution over r intervals each of size h, we can compute, for the mid-point x_i of each interval, the quantities

$$t_i = \frac{(x_i - \bar{x})}{\sigma} \quad , \quad P_i = \frac{h}{\sigma\sqrt{2\pi}} e^{-t_i^2/2} . \tag{17}$$

The cumulative distribution also reveals the number of points in each interval N_i. If the total number of points is N_T, we now form the sum

$$\chi^2 = \sum_{i=1}^{r} (N_i - N_T P_i)^2 / N_T P_i \tag{18}$$

and from it the quantity

$$t_\chi = \sqrt{2\chi^2} - \sqrt{2r-7} . \tag{19}$$

Either χ^2 or t_χ can be used to determine the probability that the sample was taken from a normally distributed population. If a large number of sample sets are taken, and χ^2 computer for each set, it will be found that χ^2 is distributed according to a χ^2 distribution** with parameter m. This latter is the number of degrees of freedom possessed by the sample and is equal to r - 1 -b where b is the number of distribution parameters computed from the sample. Here b = 2 so that m = r - 3,

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* Burington and May, p.180.

** Ibid., p. 141, Table XIV.

or three less than the number of intervals. If m is large enough, say $m > 30$, then t_χ is distributed normally about zero mean with unit standard deviation.

To determine to what degree of confidence we can state that a set of sample points comes from a normal distribution, we compute $\bar{x}$ and σ, select r, and find χ^2 and t_χ. Let us assume for the moment that $m \leq 30$. Then the value of χ^2 can be compared to a table such as Table XIV in Burington and May.* For example if $m = 20$ and $\chi^2 = 37.566$, then the table states that $\epsilon = .01$. This means the following: if the sample set from which χ^2 was computed actually came from a normally distributed population, then only 1% of the time could χ^2 be expected to be greater or equal to 37.566 for $m = 20$. One would be tempted to say, then, that this sample has a low probability of coming from a normal population. Usually a level is chosen such as $\epsilon = .05$, .01, or .001 such that the statement: "within a confidence level of 5%, the given sample does not come from a normally distributed population" can be made. Take another example, say $m = 15$, $\chi^2 = 8$. Such a sample almost definitely comes from a normal population since over 90% of the samples from this population will have values of $\chi^2 \geq 8$ for $m = 15$.

When $m > 30$, the χ^2 distribution become normal so that t_χ is the useful parameter. This value is compared in a similar manner to a table such as Table III in this memorandum.** Again, if the sample from which t_χ is computed actually comes from a normal population, then a certain fraction ϵ of the time, $t > t_\chi$. If this fraction is low,

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* Ibid, p. 286.

** Table III is compiled from Table IX of Burington and May by simple additions or subtractions.

Table III: Table of Values of t for $\frac{1}{\sqrt{2\pi}} \int_t^{\infty} e^{-\tau^2/2} d\tau = \epsilon$

ε	t	ε	t
.9987	-3.0	.4602	+0.1
.9974	-2.8	.4207	.2
.9953	-2.6	.3821	.3
.9918	-2.4	.3446	.4
.9861	-2.2	.3085	.5
.9773	-2.0	.2742	.6
.9713	-1.9	.2420	.7
.9641	-1.8	.2119	.8
.9554	-1.7	.1841	.9
.9452	-1.6	.1587	1.0
.9332	-1.5	.1357	1.1
.9192	-1.4	.1151	1.2
.9032	-1.3	.0968	1.3
.8849	-1.2	.0808	1.4
.8643	-1.1	.0668	1.5
.8413	-1.0	.0548	1.6
.8159	- .9	.0446	1.7
.7881	- .8	.0359	1.8
.7580	- .7	.0287	1.9
.7258	- .6	.0227	2.0
.6915	- .5	.0139	2.2
.6554	- .4	.0082	2.4
.6179	- .3	.0047	2.6
.5793	- .2	.0026	2.8
.5398	- .1	.0012	+3.0
.5000	0.0		

the sample itself is certainly not very normal and has a low probability of coming from a normal population. In a manner similar to the χ^2 case, confidence levels of 5%, 1% or .1% are usually chosen as decision points. To take an example, let $m = 56$, $t_\chi = 2$. Then $\varepsilon \sim .023$ means that the particular sample only has about a 2% chance of coming from a normal population and thus is rejected on a 5% level but accepted on a 1% level.